On the geometrical background of Dionysius Thrax' definition of comparatives

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The article analyzes Dionysius Thrax' definition of the comparative, according to which the comparative introduces a comparison of one object with another one of the same kind or of one object with more objects of a different kind; this definition, which is apparently contraddicted by the actual linguistic usage of Old Greek, has always presented hermeneutic difficulties to both ancient and contemporary commentators and historians of grammatical thought. None of the explanations offered so far is convincing, as they limit themselves to judge the definition unsound and careless, without trying to read it inside the possible theoretical frame of conceptual reference, within which it might have been formed.

Two hypotheses are proposed. The first is that a definition of the comparative could have been introduced into the realm of technical grammar only after the establishment of a philosophical 'metatheory' of comparison around the controversies which accompanied the definition of 'relation' in the Platonic-Aristotelean tradition. The second is that Dionysius' definition is shaped within a geometrical frame of reference, linking the definition of the comparative to the geometrical, Eudoxean and Euclidean, definition of 'ratio' (*logos*) as "a sort of relation in respect of size between *two* magnitudes *of the same kind*", which closely recalls Dionysius' definition and is mainly based on the restriction (*diorismos*) that the magnitudes forming a ratio must be homogeneous*.

1. Dionysius Thrax' definition of the comparative has always presented hermeneutic difficulties to both ancient and contemporary commentators and historians of grammatical thought.

This definition reads as follows:

Συγκριτικὸν δέ ἐστι τὸ τὴν σύγκρισιν ἔχον ἑνὸς πρὸς ἕνα ὁμοιογενῆ, ὡς Ἀχιλλεὺς ἀνδρειότερος Αἴαντος, ἢ ἑνὸς πρὸς πολλοὺς ἑτερογενεῖς, ὡς Ἀχιλλεὺς ἀνδρειότερος τῶν Τρώων.

"The comparative is the noun [Dionysius does not distinguish between nouns and adjectives] introducing a comparison of one object with another one of the same kind, as 'Achilles is braver than Ajax', or of one object to more objects of a different kind, as 'Achilles is braver than the Trojans'" (Uhlig: 1883:27).

There is a basic difference between ancient and modern inter-

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preters. The authors of the *scholia* to the *Tékhne* implicitly accept the above definition as a sound one, and limit themselves to explain and enlarge it, making explicit many points which the extremely synthetical form of the definition leaves implicit. From this point of view, the *scholia* are important not only as an integration to Dionysius' brachylogical definition, but also (in a way, mainly) because, together with the text of the *Tékhne*, they build up the first sketch of what can be called a 'discourse on the comparative' in Western grammatical tradition; this initial discourse can be assembled and reconstructed through the *disiecta membra* offered by the *Scholia*, and integrated with passages by philosophers dealing with a 'metatheory' of the comparative.

Modern interpreters tend to criticize and accuse Dionysius' definition of inadequacy. The editor of the *Tékhne* for the *Corpus Grammaticorum Graecorum*, Uhlig, judges the definition as "perversam", because "peccasse autem ita technographus mihi videtur, quia ea tantum exempla, quae subigit, in fingendo ὅρῳ respiciebat" (p. LXXXV). Di Benedetto (1958:101), on the other hand, blames the definition for its "inesattezza" and uses it as a further witness of the many faults of the *Tékhne*, which, in his view, should enable us to deny its authenticity and adscribe it to a later, modest compilator rather than to Dionysius Thrax. As to the last editor, Lallot (1989), author of an outstanding and exhaustive comment to the *Tékhne*, he views this definition as incomplete and adds that the *Scholia* are ingenious, but are unable to explain the limits of the definition itself.

What all modern interpreters object to in the author of this definition (whether he is Dionysius himself or another compiler is not an important point in our present discourse), is that he is unable, or has chosen not to, base it on the actual linguistic usage of Greek (and, for that matter, of the other languages whose comparative is typologically similar to Greek, as all Western languages are). Obviously, the two main points of the definition liable to this criticism are the limitation of comparisons in the one-to-one case and in the homogeneous-objects one: it is difficult to see why comparative sentences such as "Achilles is braver than Ajax and Agamenon" (with a one-to-two comparison) or "Achilles is braver than Hector" (where the two objects belong to two different kinds, the Greeks and the Trojans) should not be perfectly grammatical and well formed in Greek as well as in any other language; neither is one satisfied, of course, by the explanation of a scholiast (*Schol. Vatic., cod. C*, Hilgard 1901:225, 14 ff.), who, attributing Dionysius' rule to his "philellenism", leads himself to Lallot's reproach that it is an insufficient and naive explanation.

1.2. As it is fairly evident, noone can deny that the definition does not conform to the actual usage of the language and that, from this point of view, it is clearly inadequate, at least according to OUR contemporary standards of inadequacy of a grammatical definition, as stemming from the postsaussurean requisite of a linguistic science based on *la langue envisagée en elle-même et pour elle-même*. To this objection a counterobjection can be advanced: did this epistemological and theoretical prerequisite hold for the premodern study of grammar and languages and, especially, for Greek grammarians?

Most of all: from an historiographic (or even better, as I prefer to say, from an 'archaeological and genealogical' - obviously *à la* Nietzsche-Foucault) point of view, the fact that a grammatical theory or definition does not conform to, nor is based on, actual linguistic usage does not seem to me to be, *in itself*, a strong and compelling reason for denying it any originality and theoretical insight into the object defined. One should rather try to contextualize it, to detect the kind of *discourse*, in which it belongs, and to read it within the cultural background and the "climate of opinion", in which it was formed, developed and put forward, until it became generally accepted and paradigmatic. Unless we can prove that this definition was wrong, inadequate, or inexplicable WITHIN Greek grammatical culture, my impression is that we do not have enough grounds to condemn it.

It is my intention to propose a hypothesis which will show that Dionysius' definition is perfectly coherent and understandable (which does not imply, of course, that we have to accept it as a satisfactory definition of the comparative by our own standards) from the viewpoint of the procedures of Greek grammarians. My hypothesis will determine the origins of this definition as stemming from some easily detectable areas of Greek culture, which were not limited to the field of grammar, but were so important as to influence writers of grammatical treatises, who were not yet bound to the concept of *la langue en elle-même et pour elle-même*.

2. A preliminary remark can be useful. It is a fact that the comparative was brought to the attention of grammarians only at a relatively late period, and with a noteable delay, compared to other grammatical categories.

In the age of the Sophists, and up to the period of Stoic theorization on grammar, nearly all main grammatical categories had been recognized and defined by philosophers, who were interested in language: Protagoras, the theoretician of correctedness of speech (ὀρθοέπεια), proposed a division of *lógos* into four kinds (or seven, according to another source), but we also know that the division of *lógos* was a widespread object of research (D.L. IX 54 = 80 A 1 D.-K.; cp. also 80 A 25); he divided grammatical gender into the three species: masculine, neuter, and feminine (Arist., *Rh.* 3, 1407 b 6 = 80 A 27), a division whose popularity is witnessed by Aristophanes' parody in the *Clouds*; Diogenes Laertius (IX 52) reports that Protagoras had been the first to divide the parts of time (μέρη τοῦ χρόνου), a statement which is highly probable to be referred to verb tenses, as a division of physical time would not have been viewed as a novelty. Lastly, he is certain to have theorized about modalities, if not about moods, of verb, as is shown from his famous criticism of Homer for employing an imperative referring to a goddess (80 A 29).

The distinction between noun and verb was given the status of a semantic definition in Plato's *Sophist* in the context of a theory about the genesis of *lógos*, before being formalized and made canonical in Aristotle's *de interpretatione*.

As compared to this abundance of grammatical systematizations, a theory of the comparative had to wait until Dionysius Thrax, in whose *Tékhne* the first definition of this category is to be found. This fact cannot but look peculiar, if we consider that a 'comparative point of view' was one of the most characteristic features of Greek thought, especially, but not only, in its philosophical side, starting, at least, with Anaxagoras, who sees in the activity of συγκρίνειν one of the functions of the νοῦς, together with the opposite one of διακρίνειν and ἀποκρίνειν.

Of course, this delay can be due to mere chance or to lack of explicit evidence: after all the mention of the comparative in the grammatical papyrus *P.Heid. Siegman 198*, n. 12 Wouters, (whose chronological relation to the *Tékhne* is difficult to determine, due to the doubts about the latter's attribution to Dionysius) and the *scholia* to the *Tékhne* witness the working out of a theory of comparative more gradual and widespread than would result from the brachylogical definition of Dionysius. Therefore, we can suppose that Dionysius' definition was not totally original, but was rather the culmination of an earlier process of theoretical meditation.

On the other hand, a whole series of historiographical considerations allows us to suppose that this delay was not totally due to chance and that, quite the contrary, it is the signifier of an initial uncertainty which brought about a postponment of the theoretical

prerequisites necessary for the definitory and functional elaboration of this fundamental category.

2.1. It is not too difficult to assume that grammarians and philosophers were aware of the existence of the comparative as a distinct element of grammar *before* Dionysius Thrax introduced it into grammatical technical treatises; this can be witnessed, for instance, by Plato's reference, in the *Philebus*, to the two Forms of More and Less (τὸ μᾶλλον καὶ τὸ ἧττον), as resulting from the second ontological principle, the Great-and-Small (τὸ μέγα καὶ μικρόν), which Plato exemplifies with the very use of comparatives ("hotter and colder" and so on: 22 C ff.). Therefore, the *idea* of the comparative has to be traced back at least to Plato's time, but in an implicit way it was already implied in Parmenides' denegation of those parts of ontology where Being should be qualified in terms of More and Less, rather than in those of the Plenitude of Being (οὐδέ τι τῇ μᾶλλον, τό κεν εἴργοι μιν συνέχεσθαι, οὐδέ τι χειρότερον, πᾶν δ'ἔμπλεόν ἐστιν ἐόντος: 28 B 8, 23-24). On the other hand, its *thought* and, as a consequence, its technical definition and theory could only have been possible as the result of its *problematization* (I am obviously referring to Foucault's three well known historiographical categories exposed in *Discourse and Truth*).

Before a theory of the comparative could be shaped in terms of a chapter of technical grammar, a *metatheory* of the comparative had to be worked out in the wider field of philosophy, especially within ontology and categorial logic, as well as in other more technical sectors of Greek theoretical thought, such as mathematics, geometry, physiology, medicine, etc. For reasons of space I cannot deal with this aspect presently and will therefore include it in another essay I am preparing; here I will limit myself to a short allusion to a particular aspect of this wider topic.

2.2. It is important to recall that, when Aristotle systematizes the logical (as well as ontological) category of the *relative* (πρός τι) in the *Categories* (VII, 6 a 36), he gives the comparative as his first example of this category: "'the greater' is said to be greater by reference to something outside it. For, indeed, when we call a thing 'greater', we mean by that greater *than* something" (τὸ μεῖζον τοῦθ' ὅπερ ἐστὶν ἑτέρου λέγεται. τινὸς γὰρ λέγεται μεῖζον). Therefore the comparative is, in his view, a relative concept; that is to say, a concept which "is said to be such as it is from its being *of* some other thing or, if not, from its being related to something in some other way" (Cook 1938): for this reason Aristotle is eager to underline that substance does not admit of “more and less”, that is to say of being inflected in the comparative, with some exceptions (8 a 14). This idiosynchratic rule will be inherited by later grammarians, such as, for instance, the author of the *Scholia Marciana* (*VN*) to Dionysius’ *Tékhne*, who states that comparatives are not derived from proper and common nouns (that is, from signifiers of substance), but only from adjectives: there is no comparative derived from the word “orator” (ῥήτωρ), but only from its adjective ῥητορικός, because the noun ῥήτωρ refers to the “complete orator”, τὸν τέλειον ῥήτορα (Hilgard 1901:371, 10 ff.). It is evident that this rule derives from the Aristotelean theory of the relative.

It is well known that the development of the category of relation required a long, gradual elaboration, begun by Plato in the *Sophist* (255 c-d: it has to be distinguished between those beings which are said “in themselves” – αὐτὰ καθ᾽ αὑτά – and those, which are said only “in some relation”: πρὸς ἄλλα) and continued inside the Academy, before being made canonical and formalized in the context of the foundation of formal logic in Aristotle’s *Organon*.

It is my intention to develop in another essay the connections between the development of an independent category of relation and the “battle of giants”, which opposed the socratic-platonic-academic-aristotelean line of thought to sophistic panrelativism, as expressed in Protagoras’ well known formulas “πάντ᾽ ἀληθῆ” (“all propositions are true”) and “οὐκ ἔστι ἀντιλέγειν” (“it is impossibile to contradict”). In the line of attack against the Sophists, it seemed essential to establish the ontological, as well as ethical, view of an objective and absolute Truth (ultimately going back to Parmenides’ *Alétheia* as radically opposed to *dóxa* and to merely phenomenal appearances) in opposition to the relativism implied by sophist *logology* (Cassin 1986:17). According to this view, truth is a discursive, (inter)subjective matter, built up inside each single speech act and based upon Protagoras’ principle that “what appears to me, it is-to-me”; the ultimate implicit result is that there is no Truth-in-Itself, no αὐτὸ καθ᾽ αὑτό, but only just as many true-for-someone propositions as there are different acts of speech between participants in a dialogue situation.

The relevance of this “battle of giants” (γιγαντομαχία) to the development of a (meta)theory of the comparative is to be seen in Protagoras’ radical view: “I cannot conceive that one of these men [i.e. the sick and the healthy, who have different sensations of the same object, e.g. wine] can be or ought to be made wiser than the other...these experiences which the inexperienced call true, I maintain to be only better, and not truer than others”: *Tht.* 166 E ff., Jowett 1964). To this questioning of the absolute character of truth, upon which a *comparative* view of reality ultimately rests, Aristotle felt compelled to reply in an equally radical and definitive way that “there is More and Less in nature” (τό γε μᾶλλον καὶ ἧττον ἔνεστιν ἐν τῇ φύσει: 1008 b 32): notice that Aristotle introduces this statement in order to confute the denegation (implied in Protagorean relativism) of the principle of non contradiction; in this context Aristotle also lays down the foundations of a referential semantics as opposed to the Sophists’ view of language as a *plaisir de parler* (Cassin 1986; Cassin and Narcy 1989).

As a result, the only way to defeat sophistic relativism was to accept the existence of relative as a concept and as an aspect of reality, but, at the same time, to demonstrate that relation is only *one* aspect of reality, and that, beside it, the vast and fundamental field of the In-itself, αὐτὸ καθ᾽ αὑτό, extends to grant an objective ground for the establishment of absolute Truth as well as of absolute Good. It was important to detect the relative as an independent, but limited and formally well defined, category, identified, and therefore isolated, within precise borders, πέρατα, which allow it to be distinguished from the category of the Absolute, in order to avoid any lapse into the dangerous indefiniteness (ἄπειρον) of Protagorean πάντ᾽ ἀληθῆ or Anaxagorean πᾶν ἐν παντί.

The ontologically priority of the In-itself, be it the world of Platonic Forms or Aristotelean *ousíai,* does not allow of more and less: no Form can be more or less than what it is, neither can a substance (“man”) be more or less: there is no such thing as a ‘more or less man’: the More and the Less only apply to qualities (“warmer and colder”, “quicker and slower”) and quantities (“double and half”) and are relative concepts: something can only be warmer in relation to something colder; “double” can only be such as opposed to “half”, “right” as opposed to “left”.

From this it is easy to infer that a grammatical theory of comparative could only have been established after a *meta*theory of comparative, based on the logical category of relation, had been completely developed. If the cultural battle had been won by Protagoras and the Sophists, not to mention other minor schools, which were defeated by the Platonic and/or the Aristotelean lines, it can be presumed that grammarians would have defined the comparative in an utterly different way.

3. After this digression we can go back to the problem of interpreting the far from easily understandable definition of Dionysius.

The first clue that it was coherent and justifiable inside Greek culture, comes from the fact (which has not been so far recognized) that it is confirmed by analogous definitions in the parallel field of treatises of rhetoric, where the equivalent of grammatical comparison, σύγκρισις, was accurately dealt with and defined.

"Comparison" (σύγκρισις) was a rhetorical figure mainly used in the literary genre known as *encomion*, where two objects are compared in order to establish the excellence of either of the two: Theon (Spengel 1853: II, 112 ff.) defines it as a discourse establishing the better and the worse (τὸ βέλτιον ἢ τὸ χεῖρον παριστάς). Aphthonius' definition (ib.:42) is particularly significant from our viewpoint: σύγκρισις is a "comparative discourse, which by juxtaposition concludes attributing the more or the equal to the object undergoing comparison" (λόγος ἀντεξεταστικὸς ἐκ παραθέσεως συνάγων τῷ παραβαλλομένῳ τὸ μεῖζον ἢ τὸ ἴσον): the translation omits to stress the many indications to the technical use of nearly each lexical item in sciences other than rhetoric: the verb παραβάλλειν, for instance, was the technical term referring to the well known Pythagorean-Eudoxean-Euclidean method of "applying" a figure on a line or a figure on another figure (itself a method showing several affinities with a theory of comparatives, as I intend to show elsewhere); as to ἀντεξεταστικός, it was an established technical term for the grammatical discourse about comparative, as results from its use in the *scholia* and commentaries to Dionysius, such as, e.g., the *Commentar. Byz.* (Hilgard 1901:573, 25 ff.), whose author practically quotes the rhetorical definition: "comparison is a comparative (ἀντεξεταστικός) discourse, which by the juxtaposition of another person, concludes attributing the 'more' to the object undergoing comparison"; the noun, from which it derives, ἀντεξέτασις, signifies "discrimination (of a part of speech)" in Apollonius Dyscolus (Uhlig 1910:220, 4), but it was also a technical term of arithmetics, where it referred to the comparison of an integer with the sum of the preceding integers (cp., e.g., *Theol. Ar.* 10).

What is closely connected to our analysis, is that the rhetoricians are very careful to emphasise that σύγκρισις, as a rhetorical figure, has to be introduced between *two*, and only two, objects: as Hermogenes states in his *Progymnasmata* (ib.: 14), "we compare a town to a town, ..., and a kind to a kind, or a food to a food, ..., justice and wealth, etc.": that is to say, the comparison is ἑνὸς πρὸς ἕν, as Theon: 114, puts it, with the phrasing, that Dionysius uses in his grammatical definition. According to Aphthonius (*loc. cit.*) the comparison is between beautiful and useful objects, or between base ones, also between useful and harmful, smaller and bigger objects. From these examples it also appears essential, even if it is not explicitly stated, that the comparison has to be performed between objects belonging to the same species, as two different instantiations of the species ("town to town") or as contraries and contradictories ("useful to harmful"). I wish to quote an example, which will be useful to our analysis of Dionysius' definition. Aphthonius (p. 43) deals with the case of a comparison between two objects, that are not, strictly speaking, homogeneous. The comparison between Achilles and Hector from the point of view of their virtue (ἀρετὴν ἀρετῇ συγκρῖναι), seems to be impossible according to the rules, due to the fact that the two men were from different countries, a criterion, which reminds of that implied by Dionysius and the *scholia* that the comparative only applies between two Greeks (Achilles and Ajax), but not between a Greek and a Trojan (Achilles and Hector); in spite of this, we can still compare the two men, according to Aphthonius, because this fundamental difference (an element of heterogeneity) is compensated by a whole set of similarities (homogeneity): they were both born in renowned cities, they both descend from Zeus, they were both brought up as brave men and, therefore, having received the *same* training as far as courage is concerned, they achieved the *same* glory; they showed the *same* strength in the *same* war; therefore, as they had a similar life and a similar death, they are *similar* (ὅμοιοι). We should note the recurrent use of the words "same" and "similar", two concepts which were central in the theory of comparative, as we will see; at any rate, from this argument, it appears that the criterion of homogeneity was an essential part of the rhetorical theory of comparison, as it was in the grammatical one. Another indirect proof comes from an interesting passage of Theon's *Progymnasmata* (p. 114), where he sets up the rules to follow in the case of a comparison between two sets of objects belonging to two different species, as when one would compare the two species of men and women, in order to determine which one is the more courageous. In this case, it is necessary to choose, within each of the two species, the ἀκρότατα, that is the two persons most endowed with the quality under discussion, in this case courage. A comparison between the most courageous man and the most courageous woman will extend this quality to the whole set: if Themistocles, the most courageous man, proves to be more courageous than Artemisia, the most courageous woman, it is possible to infer that the whole species of men excels in courage over the whole species of women. Another criterion is quantitative: the species, that contains a larger number of people showing the quality exceeds the other: e.g., even if Tamiris is found to be more courageous than Cyrus, this does not necessarily imply that the species of women is more courageous than that of men, since it would only lead to the conclusion that one or two women are courageous, whereas the number of courageous men is much larger.

3.1. The obvious conclusion is that the two definitions of σύγκρισις, the rhetorical and the grammatical, match and confirm each other as acceptable and coherent definitions within the conceptual background of Greek culture.

Two hypotheses are possible as a consequence of what we have just seen. On both a philological and a chronological basis, we might suppose either that the grammatical definition of comparison influenced the rhetorical one or vice versa, that is that the latter was derived from the former. It might also be presumed, of course, that both definitions were formed together within the circles of grammarians and rhetoricians, due to the strong interconnections between the two disciplines.

At a deeper level, though, I think that I can advance another hypothesis, which seems to me to be highly probable and convincing, as it would allow us to trace *both* definitions to a more general conceptual frame, which was derived from one of the leading sectors of Greek thought in the period during which we may presume that a definition, or a set of tentative definitions later abandoned in favour of the Dionysian one, were in the process of being formed. I am referring, strange as it can seem, to the field of mathematics and, especially, *geometry*.

3.2. We need to try to abandon reading Dionysius' definition with our modern, postsaussurean eyes and give ourselves the difficult task of going through it with the textual memory of a Greek reader, who was (or was expected to be) an educated person, whose *paideia* had been shaped around the disciplines and the conceptual frame required, e.g., by Plato's *Republic* and *Laws*, with philosophy and mathematical sciences forming the core of his intellectual curriculum: "No one will be allowed into the Academy, unless he is knowledgeable about geometry", as Plato wanted to be inscribed on the threshold of his school, thus hinting at the basic requirement of his didactic program.

Two important concepts derived from Foucault (1969) are relevant to an "archaeology" (not simply a history) of the discourse about comparatives.

The first has to do with the supposed unity of a text, which at closer inspection will appear as a relative and variable one, since every book "est pris dans un système de renvois à d'autres livres, d'autres textes, d'autres phrases: nœud dans un réseau"; its relative unity "ne se construit qu'à partir d'un champ complexe de discours" (Foucault 1969:34).

From this point of view, Dionysius' definition will appear as a hypertext transposing into a grammatical context a series of hypotexts derived from other disciplines, which he considered particularly appropriate to form the conceptual frame for a rational definition of the "comparative noun", συγκριτικὸν ὄνομα. The lexical items involved in the definition (and we know how important the wording of definitions was in a postaristotelean definitory procedure) acquire, in this perspective, a double function. On a merely semantic level, their meanings contribute to the general meaning of the definition; on a deeper, textual and *discursive* level, they become signifiers of the conceptual field, from which the definition is derived: in other words, they function as a sort of quotation of definitions of objects pertaining to other sciences, taken as warrants to the rationality, scientific reliability and soundness of the definition itself.

It is to be underlined, again following Foucault (1969:39), that the analysis of a discourse (in our case the discourse on comparatives) does not only imply an understanding of what has been said, but also, maybe even mainly, of what has *not* been said: "comment se fait-il que tel énoncé soit apparu et nul autre à sa place?". Dionysius' definition of the comparative might have been another one, derived from another conceptual frame (or other conceptual frames) and supported by principles of other disciplinary fields: this does not mean choosing a counterfactual historiographical perspective, but the more proficient procedure by which to understand, indirectly rather than counterfactually, why *that* particular conceptual frame was chosen, rather than (an)other one(s).

4. The ideal Greek reader of the *Tékhne*, highly educated and geometrically trained, or at least familiar with mathematical notions, whose textual memory we are trying to reconstruct, certainly recognized the striking similarity (nearly a word for word quotation) between the wording of Dionysius' definition of the comparative and the Eudoxean definition of "ratio" (*lógos*), as we know it from Euclid's *Elements*, V def. 3:

Λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ πηλικότητα ποιὰ σχέσις.

"A ratio is a sort of relation in respect of size between *two* magnitudes *of the same kind*" (Heath 1926:I, 114: my italics).

This definition had become canonical and can be found in practically all mathematical textbooks, such as Theon Smyrnaeus ("Ratio is a sort of relation of two analogycal homogeneous terms between each other": Hiller:73, 16), Nicomachus of Gerasa ("Ratio is a relation of two terms betweeen each other": *Ar.* I, XXI) and Iamblichus (*in Nic.* p. 98, 15 ff.). The definition given by Iamblichus is of particular interest, because it provides a series of clear examples:

"The proportional ratio of two homogeneous terms is a reciprocal relation between them. 'Homogeneous' is added, because it is befitting to compare (συγκρίνειν) objects falling under the same kind, such as, e. g., a mina to a talent, whose common kind is weight, a line to a surface or a solid, because their common kind is magnitude.... As to non homogeneous objects it is impossible to know how they relate to one another, as in the case of a cubit to a *kotule* [a dry measure]..."

4.1. A mathematical ratio, as defined by Eudoxus, in order to be recognized as such, must satisfy two conditions. The first is that it has to be established between *two and only two* magnitudes: if more than two magnitudes are introduced, their relationship is no more a ratio, but a (possible) proportion, i.e. a "same-ratio" relationship:

"A proportion in three terms is the least possible." (*Elem.* V def. 8)

This rule (or, rather, definition) is set up by Euclid after defining proportion:

"Let magnitudes which have the same ratio be called proportional."

(Τὰ δὲ τὸν αὐτὸν ἔχοντα λόγον μεγέθη ἀνάλογον καλείσθω: V def. 6; sameness of ratio is dealt with in def. 5, which I will quote later).

After stating this numerical rule in order to distinguish between ratios and proportions, Euclid proceeds to give a more precise definition of three- and four-term proportions:

"When three magnitudes are proportional, the first is said to have to the third the duplicate ratio of that which it has to the second" (V def. 9)

and:

"When four magnitudes are <continuously> proportional, the first is said to have to the fourth the triplicate ratio of that which it has to the second, and so on continually, whatever be the proportion" (V def. 10).

Euclid is very careful to observe this sort of numerical distinctions in his propositions. For instance, when he proves the rule which allows to find the greatest common measure for numbers in book VII and for magnitudes in book X, he first proves it for two objects:

"Given two numbers not prime to one another, to find their greatest common measure" (VII prop. 2; cp. X prop. 3 for magnitudes)

and then he extends the proof to the case of three numbers:

"Given three numbers not prime to one another, to find their greatest common measure" (VII prop. 3; cp. X prop. 4 for magnitudes)

which, as Heron had already noticed (quoted by Heath 1926:II, 302), can be used to find the greatest common measure of as many numbers as we please, because any number measuring two numbers also measures their greatest common measure; and hence we can find the greatest common measure of pairs, then the greatest common measure of pairs of these, and so on, until only two numbers are left and we find the greatest common measure of these. As Heath notes, Euclid tacitly assumes this extension in VII prop. 33 ("Given as many numbers as we please, to find the least of those which have the same ratio with them"), where he takes the greatest common measure of as many numbers as we please.

4.2. The second, and even more important, condition for the establishment of a ratio (as well as, of course, of a proportion) requires the terms to be *homogeneous*. This is one of the reasons why Euclid supplements his definition of ratio with def. 4, stating that:

Λόγον ἔχειν πρὸς ἄλληλα μεγέθη λέγεται, ἃ δύναται πολλαπλασιαζόμενα ἀλλήλων ὑπερέχειν.

"Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another".

to which the famous "eudoxean-archimedean postulate" follows:

> "Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order" (V def. 5).

4.3 A particularly clear example of the care taken by Euclid to observe this condition in his demonstrations, but also of the difficulties caused by this condition, can be seen in the two procedures followed in his proves of X prop. 5, and X prop. 9.

The well known prop. 5 states that:

> "Commensurable magnitudes have to one another the ratio which a number has to a number."

The historical and cultural importance of this theorem lies in the fact that it gives a rational, geometro-algebric, demonstration to one of the basic principles of Pythagorean as well as Platonic ontological view (shared by the great majority of Greek schools of thought) that reality is expressable in numerical terms, as number to number.

In modern terms, the argument of prop. 5 runs as follows (I am quoting from Heath 1926:II, 25).

If a, b be commensurable magnitudes, they have some common measure c, and

$$a = mc,$$
$$b = nc,$$

where m, n are integers.

It follows that $c : a = 1 : m$ ....(1)

or, inversely $a : c = m : 1$,

and also that $c : b = 1 : n$,

so that, *ex aequali* $a : b = m : n$.

As Heath observes in his commentary, in stating the proportion (1), Euclid is merely expressing the fact that a is the same multiple of c that m is of 1, resting the statement on the definition of proportion in VII def. 20, which is something like a repetition for the particular field of *numbers* of the general definition of proportion given in V def. 5 for *magnitudes*. It is a well known fact that Euclid defines propor-

tions *twice*: in book V for magnitudes and in book VII for numbers; this apparently strange procedure might be due to the fact that most mathematicians and philosophers (Aristotle was the strongest supporter of this view) were careful to distinguish *numbers* from *magnitudes*, on the basis of their view that magnitudes can be incommensurable, whereas numbers do not know the embarassing phenomenon (a logical as well as ontological scandal) of incommensurability. Therefore, as Heath continues, in the demonstration of prop. 5 the statement of the proportion is not legitimate (as c, a are not numbers, but magnitudes) unless it is proved that it is true in the sense of V def. 5 with regard to magnitudes in general, the numbers 1, m being *magnitudes*. Again in Heath's view, Euclid ought to have proved that magnitudes which are proportional in the sense of VII def. 20 are also proportional in the sense of V def. 5, or that the proportion of numbers is included in the proportion of magnitudes as a particular case.

Euclid must have been conscious of these difficulties, as can be seen by his procedure in the proof of the other theorem (X prop. 9), stating that:

> "The squares on straight lines commensurable in length have to one another the ratio which a square number has to a square number; and squares which have to one another the ratio which a square number has to a square number will also have their sides commensurable in length[...]"

This theorem is a logical consequence of the previous one, as it results from its application at the beginning of the demonstration. In modern symbolism, the proof runs as follows:

If a, b be straight lines, and, as a consequence of X prop. 5,:

$$a : b = m : n,$$

where m, n are numbers,

then $a^2 : b^2 = m^2 : n^2$;

and conversely.

But this inference was by no means so easy for Euclid owing to the fact that a, b are straight lines, and m, n numbers. He has to pass from $a : b$ to $a^2 : b^2$ by means of VI prop. 20, por. ("it can be proved in the case of quadrilaterals that they are in the duplicate ratio of the corresponding sides"): the square on a is to the square on b in the duplicate ratio of the corresponding sides a, b. On the other hand, m, n being *numbers*, it is VIII prop. 11 ("Between two square numbers there is one mean proportional number, and the square has to the square the ratio duplicate of that which the side has to the side") which has to be used to show that $m^2 : n^2$ is the ratio duplicate of $m : n$. Then, in order to establish his result, Euclid assumes that, if two ratios are equal, the ratios which are their duplicates are also equal, an assumption which is nowhere proved in the *Elements*, but can be an easy inference from V prop. 22 ("If there be any number of magnitudes whatever, and others equal to them in multitude, which taken two and two togther are in the same ratio, they will also be in the same ratio *ex aequali*"): see Heath (1926:III, 31 and II, 242 ff.).

From the point of view of the homogeneity-condition, it is important that Euclid offers two different proves, one for magnitudes, and another for numbers.

4.4. There are many instances of this procedure, by which Euclid proves that two different figures are to one another as one of their components, either their bases or their volumes and areas, proving the ratios separately for each set of homogeneous objects. A simple example will suffice.

In VI prop. 1 Euclid proves the theorem:

> "Triangles and parallelograms which are under the same height are to one another as their bases."

Given two triangles (*ABC*, *ACD*) and two parallelograms (*EC*, *CF*), constructed on the same height, Euclid begins his demonstration by proving that the two triangles are equal, providing the conditions stated in I prop. 38 through the production of the two bases in both directions; successively he refers to well known V prop. 5 in order to prove that whatever multiple the base *HC* (obtained through the production of the original base towards the left), that multiple also is the triangle *ALC* (constructed on the new, produced base, of the original triangle *ACD*), that is to say:

> If the (new) base *HC* is equal to the base *CL* (i.e. the new base towards the right), then the triangle *AHC* (on the base *HC*) is also equal to the triangle *ACL* on the base *CD*;
> If the base *HC* is in excess of the base *CL*, then the triangle *AHC* is also in excess of the triangle *ACL*;
> And if less, less.

According to V def. 5, the ratio between the equimultiples of the two bases and the two triangles allows to prove that, as the base *BC*

is to the base *CD*, so is the triangle *ABC* to the triangle *ACD*. At this point, Euclid proves that, as the parallelogram *EC* is double of the triangle *ABC* and the parallelogram *FC* is double of the triangle *ACD* (the proof has been given in I prop. 41), and as parts have the same ratio as the same multiples of them (proven in V prop. 15), it is possible to infer that:

> As the triangle *ABC* is to the triangle *ACD*, so is the parallelogram *EC* to the parallelogram *FC*.

And: having been proved during the demonstrative procedure that:

> As the base *BC* is to the base *CD*, so is the triangle *ABC* to the triangle *ACD*

and that:

> As the triangle *ABC* is to the triangle *ACD*, so is the parallelogram *EC* to the parallelogram *CF*.
>
> Therefore, also:
>
> As the base *BC* is to the base *CD*, so is the parallelogram *EC* to the parallelogram *FC*.

It can be seen that the demonstration proceeds proving successively and separately that there is a ratio between base and base, triangle and triangle, parallelogram and parallelogram, thus leading to the result that the same proportional ratio obtains between the two figures and between them and their bases. We should also notice the accuracy of the wording, which stresses the field of pertinence of the three different ratios through the repetition of the words relating to the three sets of two, homogeneous objects: the *triangle*...the *triangle*; the *parallelogram*... the *parallelogram*; the *base*... the *base*. This wording, careful to the point of risking to appear monotonous to a lay reader, witnesses the careful distinctions which Euclid presumes to be necessary in order to meet the condition of homogeneity in the establishment of ratios and proportions.

4.5. From our point of view, it is relevant to observe that the proof of a proportional ratio occurring between two or more sets of objects, makes use of the three expressions "is/are equal to", "is (are) in excess", "is less than". Such procedure will be repeated several times in the propositions based on the method of exhaustion of book XII, where the proportions between figures, volumes, and areas are proved through a *reductio ad absurdum* of the opposite hypothesis: in order to prove that the area or volume A of a given figure is equal to an area and volume B, it is proved that it is *not* the case that $A >< B$, showing that, whatever difference obtains between A and B, it will necessarily be found to be less than any arbitrarily chosen magnitude. This is an application of the "antanairetic" process proved in the fundamental theorem X prop. 1, according to which, at the end of an in(de)finite process of subtractions between two unequal magnitudes, there will be left some magnitude which will be less than any lesser magnitude set out.

The importance of this proposition was outstanding, because it allowed Eudoxus, and later Euclid and Archimedes, to go back to the Zenonean-Anaxagorean infinitesimal perspective after its challenge by the Atomists, who wanted to arrest the process of infinite division to an atomic limit. As Anaxagoras had put it, "bigness and smallness are unlimited" (59 B 1) and "within the Small there is no smallest, but only the infinitely smaller; the same holds for the Great: from this point of view the Small is equal to the Great and any object is both great and small" (τοῦ σμικροῦ οὐκ ἔστι τό γε ἐλάχιστον, ἀλλ'ἔλασσον ἀεί...ἀλλὰ καὶ μεγάλου ἀεί ἐστι μεῖζον. καὶ ἴσον ἐστὶ τῷ σμικρῷ πλῆθος, πρὸς ἑαυτὸ δὲ ἕκαστόν ἐστι καὶ μέγα καὶ σμικρόν: B 3), which meant that there is no limit to infinite division of magnitudes: the way was open to the geometrical analysis of the infinite and the infinitesimal, which will be brought to perfection through the method of exhaustion by Eudoxus, Euclid, and Archimedes. On the other hand, the typical refusal of Greek thought to admit of the infinite (the *Apeiron*, which opened the negative column of the Pythagorean table of opposites), as exemplified by Aristotle's admission of the infinite as only in potence, but never in act, did not favour the full development of an infinitesimal calculus. As historians of Greek mathematics have pointed out, the lack of the modern concept of "limit" (in the process of exhaustion, the circle, rather than being actually exhausted by the regular inscribed polygons with successively smaller sides, functions as their limit) was the cause of the adoption of a negative proof *ad absurdum* or *impossibile*, as well as of the necessity to prove each theorem based on the method of exhaustion separately, rather than building a universal theorem covering all possible cases. As Enriques and Santillana (1936:51) have pointed out, the principle of contradiction

and the *reductio ad absurdum / impossibile*, these two formidable tools of Western logic, have been brought to a formal rigour (through Zeno's arguments) as a result of the discovery of the infinitesimal and related problems: each time it is required to prove an equality (of surfaces, volumes etc.) with recourse to the infinite, it is possible to avoid "la démarche aventureuse" of summing up an infinite number of magnitudes and of bisecting a whole into an infinite number of parts by proving, with a finite number of logical passages, that *it is not the case* that the magnitudes, whose equality is presumed, are unequal, because neither is bigger than the other (on this extremely important point, cp. *inter alios*: Zeuthen 1896:64 ff.; Frank 1923:46 ff.; Tannery 1930 chapter X; Rufini 1961; Mondolfo 1935:89-145; Id. 1965:196 ff.).

The three sets of expressions employed in Euclid's exhaustion propositions, as well as the problems connected with their use, are naturally at the core of the field of the (grammatical) comparative, as well as of a metatheory of comparatives, thus showing how strong the links were between the geometrical theory of ratios and proportions and the theory of comparison: we have seen before that Iamblichus referred to "comparison" (συγκρίνειν) in his definition of *lógos*.

5. There had been anticipations of this procedure in the Pythagorean tradition.

Alexander of Aphrodisia (*in Metaph.*:38, 10 ss.), after reporting the main features of Pythagorean numerology and theory of musical harmony, gives us a sketch of their theory of a heavenly harmony. Its presupposition is that heavenly bodies move around the centre ἐν ἁρμονίᾳ and that some of them rotate quicker (θᾶττον) than others, whose rotation in slower (βραδύτερον); their movements produce a sound, which is low in the case of the slowest bodies, high in that of the quickest ones. These sounds are produced according to a ratio among distances so as to cause an ἐναρμόνιον ἦχον. For instance: given that the distance of the sun from the earth is double that of the moon, triple that of Venus, quadruple that of Mercury, and so on according to an "arithmetical ratio", the most distant bodies have a quicker rotation, whereas the nearer they are the slower they move.

It is easy to see that this theory applies three parameters: distance, speed, and the nature of the sound, according to a ratio of direct proportion; the greater the distance, the greater the speed, the greater the height of the sound. That is: + distance : + speed : + height = - distance : - speed : - height.

For example: the sun's distance from the earth is double that of